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Investment strategies and hidden variables

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Abstract. The present study shows how the information on 'hidden' market variables effects optimal investment strategies. We take the point of view of two investors, one who has access to the hidden variables and one who only knows the quotes of a given asset. Following Kelly's theory on investment strategies, the Shannon information and the doubling investment rate are quantified for both investors. Thanks to his privileged knowledge, the first investor can follow a better investment strategy. Nevertheless, the second investor can extract some of the hidden information looking at the past history of the asset variable. Unfortunately, due to the complexity of his strategy, this investor will have computational difficulties when he tries to apply it. He will than follow a simplified strategy, based only on the average sign of the last l quotes of the asset. This results have been tested with some Monte Carlo simulations.

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1 Introduction

It is well known that the value of financial assets is influenced by many variables. Information on this variables and the relation between them and the value of stocks is not always available to all investors for many reasons. Consider a variable which represents an institutional prevision about the economic trend but it is still not public or the GDP growth. Also consider an insider who has privileged information on a given stock. In the first case, while these variables are known to influence the asset value, not all investors are able to use this information in the best possible way because of their different ability in modeling the impact of this variable on the stock price. In the second case, differently from the previous example, the majority of investors will not have access at all to this extra information while only a restricted group of investors will have.

The extra information, when available, can be used by investors to improve their investment strategies and the more the investor knows on this variables the more his profit will increase.

The optimal strategy is usually defined as the one that optimizes a given utility function which takes into account risks. Due to different risk's aversion of different investors, the optimal strategy can be obtained through optimization of different utility functions. This arbitrariness can be removed if one considers that the rate of growth of the capital is an almost sure quantity in the long run. Therefore, the best strategy is the one which maximizes this rate, i.e. the one which maximizes the expected logarithm of the capital. Any other strategy ends almost surely with an exponentially in time smaller capital [1,2]. Nevertheless, it is meaningful to consider different utility functions when the investment is not protracted for a very long time and risk aversion must be taken into account.

In this work we present optimal investment strategies achievable by investors with different knowledge on market variables. Our approach is based on the optimization of the capital growth rate [1] which is strictly related to the Shannon information rate [3]. We stress that our model can be used also when other utility function are preferred in order to take into account the risk aversion of investors.

We will consider two type of idealized investors: one who can only observe the past asset quotes (ordinary investor) and one who has also access to an 'hidden' variable which varies in time and which can influence the asset value (privileged investor). Note that the second investor may not exist if the hidden variable is not observable. Both asset value and 'hidden' variable will be modeled as stochastic processes.

We will only consider binary processes for the market variables. This is done to make it more realistic and operatively applicable to real data. In fact, in the first section we will show how high frequency data from foreign exchange market can be coded into a binary process. To do so we will fix a return size d (which can be both positive and negative) and wait until d, or $-d$ is reached by the rates. Then, we obtain a random sequence of returns of the same amplitude, opposite signs and randomly spaced.

In the second and third section we quantify the Shannon information available to the two investors. We find that the difference is given by the Shannon mutual information between the two variables. Then we build

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investment strategies based on the variable past history following Kelly [1]. The difference in the doubling investment rate achievable by the two investors is also found to be the Shannon mutual information.

In the forth and fifth section we choose a model where the hidden variable is a Markov process while the asset value is a subordinated process to the first one. The Shannon information and the doubling investment rate are quantified and it is shown that the privileged investor can follow a better investment strategy which is also very simple to apply. The ordinary investor can extract some of the hidden information and he improves his strategy if he looks at a long past history of the available variables. Unfortunately, the complexity of this strategy increases exponentially with the size of past time he uses for forecasting. He will than try to follow a different strategy which, even if in principle less rewarding, can be applied easier. This strategy, which is based only on the average sign of the last l observation of the observable variable, permits to reach an high level of information with a very simple algorithm.

In the sixth section we perform some Monte Carlo simulations to investigate numerically the behavior of the different investment strategies. The results show that the privileged investor is the one that gains more and, at the same time, he does not have sampling problems to estimate the Shannon information. The ordinary investor has two possibility: he uses directly the asset value to build an investment strategy or he uses the simplified investment strategy proposed here. In the first case he would need a very big amount of data to estimate all the information, in the second case, he does not have sampling problems and then he is able to retrieve a greater amount of information. In relevant cases, his information and growth rate are only by a small percentage smaller than the ones available to the investor with all knowledge.

2 Transforming price in a binary process

We show here one possible way to code a stock price, or a foreign exchange rate, into a binary sequence of symbols [4,5]. Indicating with S_t the price of a given stock and starting at time t_0 we define the return at a generic time $t > t_0$ as

$$\ln \frac{S_t}{S_{t_0}} \tag{1}$$

and we wait until time $t = t_1$ such that the return reaches the threshold d, i.e.

$$\left| \ln \frac{S_{t_1}}{S_{t_0}} \right| \geq d. \tag{2}$$

Then we start from t_1 and iterate the process to have a sequence of returns of same size d and random sign. We can now code this sequence assigning -1 to negative returns and $+1$ to positive returns. In this way one can build a random sequence of binary symbols related to the asset value.

We stress that with this procedure some of the uncertainty is moved from price to time, i.e. the size of absolute value of returns is now fixed to d but the time between two subsequent returns is a random variable.

Note that, if one considers only the new build binary sequence of returns, all the information contained in the times between returns of the same size (see [6] and references therein) is lost. This could be relevant especially if the time value of money is important. This aspect will not be treated in this work.

The procedure described above corresponds to a patient investor who waits to update his investing strategy until a certain behavior of the market is achieved, i.e. a fluctuation of size d. In fact, betting on the next more probable symbol, for an investor, is equivalent in buying the stock (if he considers $+1$ as the more probable) and wait until the return is of size d. If the sign is positive he would have increased his invested capital of a fraction d otherwise he would loose the same amount. Instead, if he considers more favorable to bet on the symbol -1 he then has to sell short the stock and update his position only when the return on the stock has reached the threshold. The return on the capital would then be exactly as in the first case but with inverted signs.

How to chose the symbol and which portion of the capital one should invest in presence of hidden variables is the aim of the next sections.

For the sake of simplicity in the following discussion, we will not only consider the asset value as dichotomic but also the hidden variable will be considered to follow a binary process.

3 Information analysis

Let us suppose that the value of a financial asset, or some function of this value like for example the volatility, is influenced by an 'hidden' variable. In the model we will consider two investors: one who has access to the hidden variable and one who has not. The 'hidden' variable can be seen for example as information available only to insiders and then one of the investor considered in the model is an insider trader, or as a macroeconomic variable, like for example, the GDP growth of some nation, and then the difference between the investors is in their ability in modeling the influence between the two variables: one who knows how to relate them, one who does not.

We will model this scenario with two stochastic processes: σ will represent the sequence defined before (or a function of this value, just for simplicity in the following we will refer to σ as the asset value), while ε will represent the hidden stochastic process. σ can be seen as a subordinated process to ε, which will be regarded, in the following, as the fundamental process. To fix the notation let $\underline{\sigma}_l(t) = (\sigma(t), \sigma(t-1), \ldots, \sigma(t-l))$ be the values of the asset at discrete times $t, t-1, \ldots, t-l$. Analogously: $\underline{\varepsilon}_l(t) = (\varepsilon(t), \varepsilon(t-1), \ldots, \varepsilon(t-l))$. We assume discrete dynamics for σ and ε, i.e. $\sigma(t)$ and $\varepsilon(t)$ may assume only 2 values.

$P(\underline{\sigma}_l)$ is the probability model which describes the statistics of the financial asset. The conditional probability of observing sequence $\underline{\sigma}_l$ given a sequence $\underline{\varepsilon}_l$ is $P(\underline{\sigma}_l|\underline{\varepsilon}_l)$.

Furthermore $P(\underline{\sigma}_l, \underline{\varepsilon}_l) = P(\underline{\sigma}_l|\underline{\varepsilon}_l)P(\underline{\varepsilon}_l)$ is the joint probability to observe the sequences $\underline{\sigma}_l$ and $\underline{\varepsilon}_l$, and $P(\underline{\varepsilon}_l)$ is the probability of the sequence $\underline{\varepsilon}_l$. The relation between $P(\underline{\sigma}_l)$ and $P(\underline{\sigma}_l|\underline{\varepsilon}_l)$ can be written as $P(\underline{\sigma}_l) = \langle P(\underline{\sigma}_l|\underline{\varepsilon}_l)\rangle$ where $\langle\cdot\rangle$ stands for the average over the process ε, i.e. $\langle\cdot\rangle = \sum_{\underline{\varepsilon}_l} P(\underline{\varepsilon}_l)(\cdot)$.

In the following we will quantify the different Shannon information [3] available to two types of investors: the ordinary one does not have the possibility to measure ε while the privileged one does.

- *ordinary investor*: in this case only the present and past values of the asset is known, namely we only know the process σ for a fixed time window in the past, i.e. $\underline{\sigma}_l$. The information rate available to the investor is given by
$$I_l = \log 2 - h_l, \qquad (3)$$
where h_l is the Shannon entropy rate and it is defined as
$$h_l = H_l - H_{l-1} \qquad (4)$$
and H_l represents the entropy of the sequence $\underline{\sigma}_l$:
$$H_l = -\sum_{\underline{\sigma}_l} P(\underline{\sigma}_l)\log P(\underline{\sigma}_l), \qquad (5)$$
$$H_0 = 0.$$
It is worth pointing out that the information I_l is a non decreasing function of l, thus the bigger is the knowledge of the asset value in the past, the bigger is the information. Note that I_l represents the average information present in the the sequence $\underline{\sigma}_{l-1}(t)$ to predict the symbol $\sigma(t+1)$. In the next section we will see that this information can be used for an investment strategy based on the knowledge of the past history of the asset.
If one knows the full past history of the asset value (i.e. one knows $\underline{\sigma}_l$ with $l = \infty$), the information available is given by
$$I = \lim_{l\to\infty} I_l = \log 2 - \lim_{l\to\infty} h_l = \log 2 - h, \qquad (6)$$
if the limit exists (it can be proved that the limit exists for stationary stochastic processes), h is the Shannon entropy related to the stochastic process σ. In this context I represents the maximum information achievable by the investor if he does not know the hidden variable.

- *Privileged investor*: in this case the investor knows the present and whole past history of the variables σ and ε. If the investor knows precisely the value of the hidden variable, he will use a more detailed probability model, $P(\underline{\sigma}_l|\underline{\varepsilon}_l)$. In this case the information rate available to the investor is
$$\tilde{I}_l = \log 2 - \tilde{h}_l, \qquad (7)$$
where $\tilde{h}_l$ is the conditional Shannon entropy rate associated with $\underline{\sigma}_l$ given $\underline{\varepsilon}_l$. $\tilde{h}_l$ is defined as follows:
$$\tilde{h}_l = \tilde{H}_l - \tilde{H}_{l-1}, \qquad (8)$$
in the limit $\tilde{h} = \lim_{l\to\infty}\tilde{h}_l$, $\tilde{I} = \lim_{l\to\infty}\tilde{I}_l$. Also $\tilde{I}_l$ is a non decreasing function of l. $\tilde{H}_l$ represent the entropy of the sequence $\underline{\sigma}_l$ given the knowledge of the hidden stochastic process $\underline{\varepsilon}_l$:
$$\tilde{H}_l = -\left\langle \sum_{\underline{\sigma}_l} P(\underline{\sigma}_l|\underline{\varepsilon}_l)\log P(\underline{\sigma}_l|\underline{\varepsilon}_l)\right\rangle, \qquad (9)$$
$$\tilde{H}_0 = 0.$$

It is intuitive that in the second case there is much more information available: i.e. $\tilde{I}_l \geq I_l$. In fact
$$\tilde{I}_l - I_l = h_l - \tilde{h}_l = I_l(\underline{\sigma}_l;\underline{\varepsilon}_l) \geq 0, \qquad (10)$$
where $I_l(\underline{\sigma}_l;\underline{\varepsilon}_l)$ is the rate of mutual information between the processes $\underline{\sigma}_l$ and $\underline{\varepsilon}_l$. $I(\underline{\sigma}_l;\underline{\varepsilon}_l)$ is always greater than zero and equals zero if and only if the two variables are statistically independent. This means that if the hidden variable and the asset value are independent, the knowledge of the hidden variable does not increase the information in the financial asset value.

In the limit $l \to \infty$, $I_l(\underline{\sigma}_l;\underline{\varepsilon}_l) \to I(\underline{\sigma};\underline{\varepsilon})$ which is the mutual information if one knows the whole past history of both processes.

In the following section we will find a relation between the information obtained by analyzing the past history of the stochastic process and the optimal investment strategy that can be used by investors to increase their wealth in the future.

4 Investment strategies

The available information can be related to the gain obtained from repeated bets. As discussed in the introduction, we will follow Kelly's approach to investment strategies [1]. This approach considers that the rate of growth of the capital is an almost sure quantity in the long run and maximizes this rate. It turns out that his maximization corresponds to the use of a logarithmic utility function and any other strategy ends almost surely with an exponentially in time smaller capital [1,2]. A discussion on the possibility of using other utility functions will be given at the end of this section.

- *Ordinary investor*: the investor knows $\underline{\sigma}_l(t)$ and allocates a different portion $b_l(t) = b_l(\underline{\sigma}_l(t))$ of his wealth on the two possible value of the variable $\sigma(t)$ at time t, such that $b_l(t) \geq 0$ and $\sum_{\sigma(t)} b_l(t) = 1$, in the case of two symbols considered here the investor would bet $b_l(t)$ on the most probable symbol and $1 - b_l(t)$ on the less probable one. Notice that this is equivalent to betting a fraction $a_l(t) = 2b_l(t) - 1$ on the most probable symbol and leaving a fraction $2 - 2b_l(t)$ of the capital in a zero interest rate account. This is true when the return on a bet is double the invested capital but can be generalized with a change in the fractions to be invested (as can be done with leverage).

We stress that the investment strategy depends only on $\underline{\sigma}_l$ because we suppose that the variable ε is not measurable by this investor. We also stress that the portfolio strategy $\underline{b}_l$ depends on the knowledge of the past history of the variable $\sigma(t)$. It can be proved [1] that the wealth of the investor using this strategy grows exponentially as

$$W_l \simeq e^{\lambda(\underline{b}_l)l}, \tag{11}$$

where

$$\lambda(\underline{b}_l) = \log 2 + \sum_{\underline{\sigma}_l(t),\underline{\varepsilon}_l(t)} P(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t)) \log b_l(\underline{\sigma}_l(t)) = \log 2 + \sum_{\underline{\sigma}_l(t)} P(\underline{\sigma}_l(t)) \log b_l(\underline{\sigma}_l(t)) \tag{12}$$

$\lambda(\underline{b}_l)$ is the doubling investment rate. Note that it is possible to perform the average across $\underline{\varepsilon}_l(t)$ because the investment strategy $b_l(\underline{\sigma}(t))$ does not depend on $\underline{\varepsilon}_l(t)$.
The best investment strategy is given by performing an optimization of $\lambda(\underline{b}_l)$ over $\underline{b}_l$. Let $\underline{b}_l^*$ and $\lambda_l^* = \lambda(\underline{b}_l^*)$ be respectively the optimal strategy and the value of the doubling investment rate after optimization. After optimization [1] the best betting strategy is given by *proportional gambling*, i.e. $\underline{b}_l^* = P(\sigma(t)|\sigma(t-1), \ldots, \sigma(t-l))$. In this case $\lambda_l^* = I_l$, and therefore λ_l^* is a non decreasing function of l. If the investor knows the present and the whole past history of the variable $\sigma(t)$ the optimal doubling investment rate is given by

$$\lambda^* = \lim_{l\to\infty} \lambda_l^* = \log 2 - h = I. \tag{13}$$

If the limit exists, $\underline{b}^* = \lim_{l\to\infty} \underline{b}_l^*$ constitutes an *optimal investment strategy*. The knowledge of the whole time series allows the investor to choose a better investment strategy, in fact $\lambda^* \geq \lambda_l^*$. The more the investor knows on the past of the stochastic process the more he can hope to gain in the future using the investment strategy described here. To be able to apply this strategy on real data the investor needs to have access to a big amount of data. In fact, to be able to estimate the Shannon information for a sequence $\underline{\sigma}_l$ of only two symbols one would need a time series much longer than 2^l because this is the number of all possible combination of 2 symbols in a sequence of length l.

- *Privileged investor*: in this case we suppose that the variable ε is observable, the investor can than choose an investment strategy which dependents on this variable $\tilde{b}_l = b_l(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t))$. The doubling investment rate is then given by

$$\tilde{\lambda}(\underline{b}_l) = \log 2 + \sum_{\underline{\sigma}_l(t),\underline{\varepsilon}_l(t)} P(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t)) \log \tilde{b}_l(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t)) = \log 2 + \left\langle \sum_{\underline{\sigma}_l(t)} P(\underline{\sigma}_l(t)|\underline{\varepsilon}_l(t)) \log \tilde{b}_l(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t)) \right\rangle \tag{14}$$

and, after optimization of this function, the optimal investment strategy is given by $\tilde{b}_l^* = P(\sigma(t)|\sigma(t-1), \ldots, \sigma(t-l), \underline{\varepsilon}_l(t))$, in this case $\tilde{\lambda}_l^* = \tilde{I}_l$ hence also $\tilde{\lambda}_l^*$ is a non decreasing function of l.
Again if the investor knows the whole past history or $\underline{\sigma}$ and $\underline{\varepsilon}$

$$\tilde{\lambda}^* = \log 2 - \tilde{h} = \tilde{I}. \tag{15}$$

We stress that the strategy in this case depends on both $\underline{\sigma}$ and $\underline{\varepsilon}$. We have already discussed that $\tilde{I} \geq I$, hence it is clear that the knowledge of both stochastic processes, σ and ε, is certainly going to increase the investment rate.

The difference $\tilde{\lambda}_l^* - \lambda_l^* = I_l(\underline{\sigma_l}, \underline{\varepsilon_l})$ is again the Shannon mutual information. Note that in the limit, $l \to \infty$, this quantity converges to $I(\underline{\sigma}, \underline{\varepsilon})$ but it is not a monotonic function of l. Indeed, even if the variables are correlated ($I_l > 0$), it might be the case that $\tilde{I} \simeq I$. This means that the knowledge of a long history of the asset variable may allow for recovering the information attached to the hidden variable. We point out that the results obtained by Kelly [1] can be reproduced if one consider the hidden variable as deterministic.

As already mentioned in this work, the logarithmic function is only one of the possible utility functions used when building an investment strategy. One could use any other utility function and in doing so one would consider the risk aversion of the investor but it would be lost completely the relation between growing rate and entropy of the process. Furthermore, it can be shown that the ratio between the wealth growing with a strategy based on any utility function and the one based on the logarithmic function tends to zero exponentially in the long run (for a discussion on this point see [1]).

In the discussion presented here, if a different utility function is used, the privileged investor will still have a growing rate higher than the ordinary investor due to his better knowledge of the market variables. Then, all the inequalities on the growing rates found here with a logarithmic optimization will still be valid if a different optimization approach is used.

5 A two states Markov model

We will model both the hidden variable and the subordinated process with a boolean 1-step process (only -1 and 1 permitted). The variable $\varepsilon(t)$ is Markovian and the probability that it does not change its value from instant t to instant $t+1$ is q and the probability that the variable switches to the other value is $1-q$. It can be proved easily that, asymptotically, $P(\varepsilon(t) = -1) = P(\varepsilon(t) = 1) = 1/2$.

The dynamics for $\sigma(t)$ is modeled starting from the value of $\varepsilon(t)$. The probability that at instant t the random variables $\sigma(t)$ and $\varepsilon(t)$ assume the same value is p; the probability that they are different is $1-p$. It turns out that, asymptotically, $P(\sigma(t) = -1) = P(\sigma(t) = 1) = 1/2$. We stress that while $\underline{\varepsilon}$ and the joint $(\underline{\varepsilon}, \underline{\sigma})$ are 1-step Markov processes $\underline{\sigma}$ is not Markovian.

The probability of a sequence $(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t))$ is

$$P(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t)) = \frac{1}{2} \prod_{i=t-l+1}^{t} P(\sigma_i|\varepsilon_i)P(\varepsilon_i|\varepsilon_{i-1}), \quad (16)$$

where

$$P(\sigma_i|\varepsilon_i) = \frac{1 + (2p-1)\sigma_i\varepsilon_i}{2}$$
$$P(\varepsilon_i|\varepsilon_{i-1}) = \frac{1 + (2q-1)\varepsilon_i\varepsilon_{i-1}}{2}. \quad (17)$$

We are using $\varepsilon_t = \varepsilon(t)$ and $\sigma_t = \sigma(t)$ according to convenience. The processes $\underline{\sigma}$ and $\underline{\varepsilon}$ are stationary hence their Shannon entropies are well defined quantities. Considering that $(\underline{\varepsilon}, \underline{\sigma})$ is a Markov process the entropy $\tilde{H}_l$, defined as:

$$\tilde{H}_l = -\left\langle \sum_{\underline{\sigma}_l} P(\underline{\sigma}_l|\underline{\varepsilon}_l) \log P(\underline{\sigma}_l|\underline{\varepsilon}_l) \right\rangle \quad (18)$$

can be calculated explicitly [7] and is given by

$$\tilde{H}_l = -l(p\log p + (1-p)\log(1-p)) \quad (19)$$

for all $l \geq 1$. Then the information is

$$\tilde{I} = \tilde{I}_l = \log 2 - \tilde{h} = \log 2 + p\log p + (1-p)\log(1-p). \quad (20)$$

for $l \geq 1$.

Note that the information (20) does not depend on the length l of the sequence $(\underline{\varepsilon}_l, \underline{\sigma}_l)$ but only on the transition probability p. If the investor knows $\varepsilon(t)$ he does not need any more information to prepare his strategy. In fact, $b_l = P(\sigma(t)|\sigma(t-1), \ldots, \sigma(t-l), \underline{\varepsilon}_l(t)) = P(\sigma(t)|\varepsilon(t))$.

If the investor has only knowledge of the process $\underline{\sigma}_l$ the entropy is given by

$$H_l = -\sum_{\underline{\sigma}_l} P(\underline{\sigma}_l) \log P(\underline{\sigma}_l), \quad (21)$$

and the information

$$I_l = \log 2 - h_l. \quad (22)$$

In this case I_l is a monotonic non decreasing function of l and has $\tilde{I}$ as its upper bound, in fact $\tilde{I} > I_l, \forall l$. Investors can build optimal investment strategies according to their knowledge of the market variables. After optimization the doubling investment rates are given by:

$$\lambda_l^* = I_l$$
$$\tilde{\lambda}^* = \tilde{I}. \quad (23)$$

While for the privileged investor it will be easy to follow his strategy, for the ordinary investor it will not be the same. He will have to use a sequence of length l comparable with the correlation characteristic length of the fundamental process. But the response space of his strategy is proportional to 2^l which can be, in real cases, a huge number.

6 A suboptimal betting strategy

Let us consider the ordinary investor in the simple scenario of the Markov model described before. He only has information on σ, then he would have to find the best investment strategy only looking at the sequence $\underline{\sigma}_l$. Nevertheless, extracting information from a sequence $\underline{\sigma}_l$ is not always possible as we can show with an example.

Consider the volatility of our asset as the 'hidden' variable and the asset price as the subordinate process. We know that asset's volatility presents long range correlation, and then a long range trend, of the order of 100 business days. We can then say that there is a characteristic time for the hidden variable of the same length. For a dichotomic process the strategy has to look, at any time step, at a previous sequence of length l. Since the response space dimension is 2^l we would need to use 2^{100} values of the probabilities $P(\underline{\sigma}_l)$. A length of the same order would be needed if one consider the 'hidden' variable as a macroeconomic trend and the subordinate process as the volatility of the asset which could be digitalized as period of high and low volatility (only two values). It is obvious that this huge number of probabilities makes it impossible to build a strategy based on them.

To solve this problem we propose a different betting strategy. Let us define the following stochastic variable:

$$\gamma_l(t) = \begin{cases} 1 & \text{if } \sum_{i=t-l}^{t-1} \sigma_i > 0 \\ \sigma(t-1) & \text{if } \sum_{i=t-l}^{t-1} \sigma_i = 0 \\ -1 & \text{if } \sum_{i=t-l}^{t-1} \sigma_i < 0. \end{cases} \quad (24)$$

This definition of γ_l gives an indication of the predominance of positive (negative) signs in a sequence of length l. Note that the variable γ_l can only take two symbols $(-1, 1)$ and the number of symbols does not depend on the length l while the sequence $\underline{\sigma}_l$, from which γ_l is defined, can take 2^l symbols. Given the symmetry of the model, one finds

$$P(\gamma_l(t) = 1) = P(\gamma_l(t) = -1) = \frac{1}{2} \quad \forall l, t \quad (25)$$

while

$$P(\sigma(t), \gamma_l(t)) = \sum_{\underline{\varepsilon}_l(t), \underline{\sigma}_{l-1}(t-1)} P(\underline{\sigma}_l(t), \underline{\varepsilon}_l(t))\theta(\gamma_t). \quad (26)$$

Then instead of looking for the best strategy among $b(\underline{\sigma}_l(t))$ we consider an optimization on the reduced space $(\sigma(t), \gamma_l(t))$. Equation (26) can be summed analytically as shown in the appendix. Then we choose different strategies according only to the value of $\gamma_l(t)$ and the present value of $\sigma(t)$:

$$b(\sigma(t), \gamma_l(t)) = A + B\sigma(t)\gamma_l(t). \quad (27)$$

The strategy has this simple form because $\sigma(t)$ and $\gamma_l(t)$ are both dichotomic. The doubling investment rate is then

$$\hat{\lambda}_l = \log 2 + \sum_{\underline{\sigma}_l(t)} P(\underline{\sigma}_l(t)) \log b(\sigma(t), \gamma_l(t)). \quad (28)$$

If one performs an optimization respect to $\underline{b}$, one finds

$$\hat{\lambda}_l^* = \log 2 - H(\sigma(t)|\gamma_l(t)), \quad (29)$$

where

$$H(\sigma(t)|\gamma_l(t)) = -\sum_{\sigma(t),\gamma_l(t)} P(\sigma(t), \gamma_l(t)) \log P(\sigma(t)|\gamma_l(t)). \quad (30)$$

Following the results in the appendix this equation can be calculated explicitly. Again the optimal choice for the portfolio is given by proportional gambling: $b^*(\sigma(t), \gamma_l(t)) = P(\sigma(t)|\gamma_l(t))$. It is obvious that $\hat{\lambda}_l^* \leq \lambda_l^* \leq \tilde{\lambda}^*$ and this relation holds also if one performs an optimization on l: $\max_l \hat{\lambda}_l^* \equiv \hat{\lambda}_{\hat{l}}^* \leq \lambda_{\hat{l}}^* \leq \lambda^* \leq \tilde{\lambda}^*$.

Note that $\hat{\lambda}_l^*$ is not a monotonic function of l given that $\gamma_l(t)$ is essentially an average over $\underline{\sigma}_l$ and then an increment in l does not necessarily increase the available information. In fact, the existence of a value of l that maximize $\hat{\lambda}_l^*$ is expected. The meaning of this strategy is that we try to guess the value of the hidden variable, $\varepsilon(t)$, by looking at the predominance of plus or minus signs in $\underline{\sigma}_l(t)$. We expect that if l is shorter than the characteristic time of the hidden variable but sufficiently long that $\gamma_l(t)$ is computed with an high number of asset values, one reaches the maximum profit.

If the investors risk aversion is taken into account, and then a different utility function is used, one can always find a suboptimal betting strategy to simplify the investment making it operatively possible for the investor to follow it.

7 Numerical results

To compare the different investment strategies we performed several Monte Carlo simulations of the Markov process, $(\underline{\sigma}, \underline{\varepsilon})$, generated according to the rules described in Section 5. We used different values for p and q and the length of the series was 2×10^5 time steps.

In the first set of simulations (results shown in Fig. 1) we investigated the behavior of the doubling investment rate estimated as a function of l for three different investors: one who has access to both $\underline{\sigma}_l$ and $\underline{\varepsilon}_l$ ($\tilde{\lambda}_l^* = \tilde{\lambda}^*$ does not depend on l), one who only knows $\underline{\sigma}_l$ (with rate λ_l^*), and the last one who also has only access to $\underline{\sigma}_l$ but he uses the simplified strategy proposed in Section 6 (with rate $\hat{\lambda}_l^*$). The results of the simulation show, as expected, that the investor with the best strategy is the privileged one. It should also be noticed that, given that this strategy does not depend on l, it is also very simple to implement since he only has to look at $\varepsilon(t)$ and then he can bet.

The interesting point are the results shown for the strategies of the ordinary investor. He can try to build a strategy based on σ but, as noticed before, the complexity of this strategy grows exponentially with l and he will not be able reach the optimal value of l.

Furthermore, in real cases, the investor does not know the 2^l probabilities $P(\underline{\sigma}_l)$, therefore he has to estimate them from a sequence of length at least $L \gg 2^l$. While,

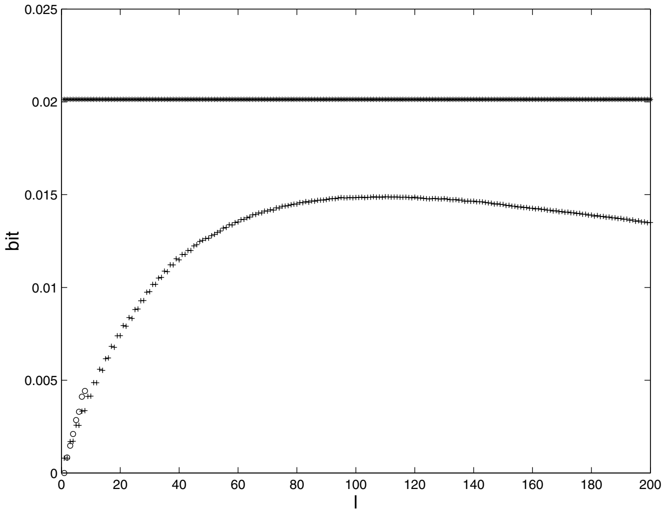

Fig. 1. Doubling investment rate as a function of l with $q = 0.999$ and $p = 0.6$, where crosses are for $\hat{\lambda}_l^*$, circles for λ_l^* and stars for $\tilde{\lambda}^* = \log 2 + p \log p + (1 - p) \log(1 - p) = 0.0201$. As explained in the text, due to sampling problems, it was not possible to estimate λ_l^* for l greater than 8.

the privileged investor, only has to estimate 4 probabilities $P(\sigma(t)|\varepsilon(t))$ and therefore he needs a number of data $L \gg 1$.

Results are shown in figure 1 with the circles, even with $L = 2 \times 10^5$ we were not able to go beyond $l = 8$ while for the privileged investor we were able to compute easily the rate for any l. The growth rate reached by the ordinary investor is only a small percentage of the one reached by the privileged investor.

A completely different result was instead obtained when the ordinary investor follows the simplified strategy based on the average sign of $\underline{\sigma}_l$. Indeed also this strategy depends on l (as shown by the crosses in the plot) but its complexity does not. In fact, the number of data needed to estimate the probabilities increases with l but only as $L \gg l$. As one can see in Figure 1, in this case we were able to compute the rate for much more values of l.

It can also be seen that this strategy is not monotone in l as the last two, in fact, there is a maximum in $\hat{\lambda}_l^*$ at a finite value of $\hat{l}$. The growth rate reached with this strategy is only by a small percentage smaller than the one of the investor with full knowledge. To summarize, the investor with partial knowledge on asset variable will be able to reach better profits if he follows a simplified strategy.

In the second set of simulations (shown in Fig. 2) we have studied the behavior of $\hat{l}$ as a function of q when $p = 0.6$. The length $\hat{l}(q)$ that maximizes $\hat{\lambda}_l^*$ is compared with the auto-correlation characteristic length, τ, of ε which can be calculated directly from equation (17) and it is given by:

$$\tau = -\frac{1}{\log(2q - 1)}. \quad (31)$$

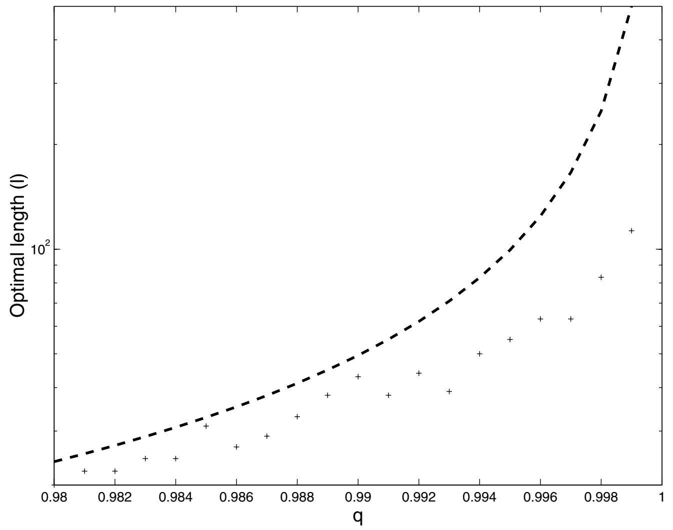

Fig. 2. Optimal length l as a function of probability q when $p = 0.6$. Crosses are for the optimal length $\hat{l}$ that maximize $\hat{\lambda}_l^*$ while the dashed line is for the autocorrelation characteristic length τ.

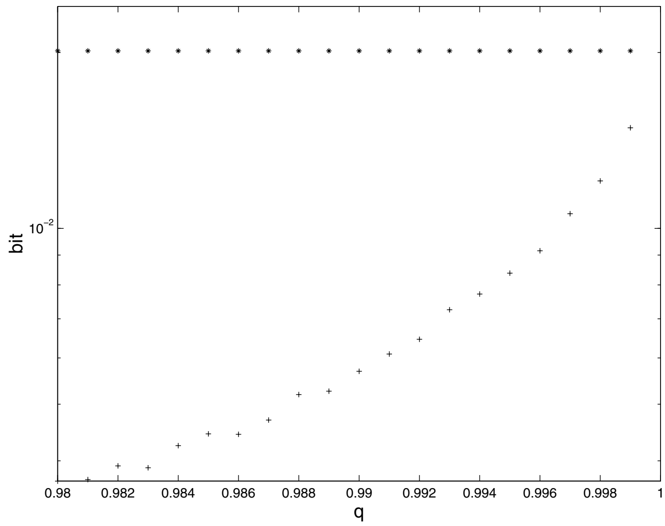

Fig. 3. Best doubling investment rate as a function of probability q when $p = 0.6$, where crosses are for $\hat{\lambda}_l^*$ and stars for $\tilde{\lambda}_l^*$.

When $q = 1 - \alpha, \alpha \ll 1, \tau \approx 1/\alpha$. The plot shows that the longer is the autocorrelation for the variable ε, the longer is the optimal length. Note that $\hat{l}$ is always smaller than τ in fact, as we already discussed, to build the variable $\gamma_l(t)$ we have to average over a big number of asset values but this number must me shorter than τ to guess better the value of $\varepsilon(t)$.

In the last set of simulations we studied $\hat{\lambda}_{\hat{l}}^*$ as a function of q when p is fixed (shown in Fig. 3 for $p = 0.6$) and compared with $\tilde{\lambda}^*$ which depends only on p as shown in the previous paragraph. The plot shows that the stronger is the autocorrelation for the variable ε the higher is the profit the investor using the simplified strategy can reach. This result can be explained easily if one consider that the stronger is autocorrelation the longer is the persistence of the same symbol (-1 or $+1$) for this variable. But, if the variable ε seldom changes its value, it will be easily predictable using the variable γ and the strategy $\hat{\lambda}_l^*$ will be more efficient.

8 Conclusions

In this work we have studied the behavior of the doubling investment rate for investors with different knowledge on a subordinate stochastic process related to the asset value and on a hidden variable influencing this process. We did not make any particular assumption on the hidden variable or on the subordinate process, the hidden variable can in fact represent an hidden trend which may not be measurable by investors or a variable known only by insiders, the subordinate process can be a stock price return, the volatility of the stock or a function of them. We considered two type of investors: one who has access to the hidden variable (privileged) and one who has not (ordinary). We modeled the hidden variable and the asset value as correlated stochastic processes. Using log optimal investment strategies we quantified the different gains achievable by the two investors. We have shown that the difference in the two doubling investment rate is given by the Shannon mutual information between the two stochastic processes.

In our model we did not take into account the efficiency/inefficiency of the market. In fact the market can be consider efficient for the ordinary investor, who has no access to the hidden variable, in this case he would make no profit from his strategy. The market can become instead inefficient if one consider the possibility that there exist a privileged investor who has access to the hidden variable.

In the second part of this work we modeled the hidden variable as a two state Markov process, the asset value was modeled as a stochastic process correlated to the first one: the joint process, asset value and hidden variable, was still a Markov process.

We have then shown that the privileged investor is not only the one that reaches higher profit but he can do that by using a very simple strategy. Instead, the complexity of the best strategy accessible to the ordinary investor is such that he cannot use it in practice. Therefore, he tries a simplified strategy easier to follow but, at the same time, able to capture most of the hidden information. This strategy is based on the average sign of the last l observation of the asset value. The response space is reduced from a dimension 2^l to 4 and then its complexity is comparable with the strategy used by the privileged investor. Furthermore, if the length l is chosen optimally, the ordinary investor is also able to capture most of the hidden information and then his capital growth rate is only by a small percentage smaller than the growth rate of the privileged investor.

Numerical results obtained through Monte Carlo simulations confirm that it is almost impossible for the ordinary investor to follow a strategy based directly on the asset value, while the simplified strategy allows him to capture easily most of the information present in the hidden variable.

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Appendix

We will show in this appendix how to find an explicit expression for $P(\sigma(t),\gamma_l(t))$:

$$P(\sigma(t),\gamma_l(t)) = \sum_{\underline{\varepsilon}_l(t),\underline{\sigma}_{l-1}(t-1)} P(\underline{\sigma}_l(t),\underline{\varepsilon}_l(t))\theta(\gamma_l(t)) \quad (32)$$

$$= \frac{1}{2}\sum_{\underline{\varepsilon}_l(t),\underline{\sigma}_{l-1}(t-1)} \frac{1+(2p-1)\sigma_t\varepsilon_t}{2} \times \prod_{i=t-l}^{t-1}\left(\frac{1+(2p-1)\sigma_i\varepsilon_i}{2}\frac{1+(2q-1)\varepsilon_i\varepsilon_{i+1}}{2}\right)\theta(\gamma_l(t)) \quad (33)$$

where we are considering $\gamma_l(t)$ as given by $\sum_{i=t-l}^{t-2}\sigma_i + \alpha\sigma_{t-1}$ with $\alpha = 1+1/e$ and

$$\theta(\gamma_l(t)) = \int_0^\infty \delta(z-\gamma_l(t))dz = \frac{1}{2\pi}\int_0^\infty dz\int_{-\infty}^\infty e^{i(z-\gamma_l(t))x}dx = \frac{1}{2\pi}\int_0^\infty dz\int_{-\infty}^\infty dx e^{izx}e^{-ix\sigma_{t-1}}\prod_{j=t-l}^{t-2}e^{-ix\sigma_j} \quad (34)$$

is the step function that select only positive $\gamma_l(t)$ (or negative for $\theta(-\gamma_l(t))$). Using this definition of θ into equation (26) and performing the summation over all possible combination of $\underline{\sigma}_{l-1}(t-1)$ one gets

$$P(\sigma(t),\gamma_l(t)) = \frac{1}{4\pi}\int_0^\infty dz\int_{-\infty}^\infty dx e^{izx}\sum_{\underline{\varepsilon}_l(t)}\frac{1+(2p-1)\sigma_t\varepsilon_t}{2} \times\left(\frac{1+(2q-1)\varepsilon_t\varepsilon_{t-1}}{2}\left(\cos(\alpha x)+i(2p-1)\sin(\alpha x)\varepsilon_{t-1}\right)\right) \times\prod_{i=t-l}^{t-2}\left(\frac{1+(2q-1)\varepsilon_{i+1}\varepsilon_i}{2}\left(\cos(x)+i(2p-1)\sin(x)\varepsilon_i\right)\right) \quad (35)$$

equation (35) can be summed over $\underline{\varepsilon}_l(t)$ analytically and then integrated numerically once p and q are chosen.

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